**Black Scholes PDE**

We have delta hedged portfolio (P) of 2 positions:

* short position in one Option (V)
* and long position in shares

We can notice that term has cancelled which means that stochastic term is removed (uncertainty is removed) only deterministic terms remains. Therefore, the portfolio must yield return to other riskless instruments, otherwise there will be an arbitrage opportunity. Therefore:

We have Black Scholes PDE:

**Finite Difference Scheme**

**Forward Difference**

* For time derivative
* For first spot derivative use central difference
* For second spot derivative use

Moreover

* dS = ∆S
* dt = ∆t
* M∆S = S\_max
* N∆t = T
* V(i,j) denotes V(i∆t, j∆S)

BS PDE

Note, that this becomes the Backward Euler using the Forward Difference which can be confusing, to say the least! This is also called Implicit Euler, since the formula for does not have explicit solution, i.e. need to use matrix algebra to solve at each time step. The Implicit method is more stable!

**Boundary Conditions**

V(n,j) = max(j∆S – K ,0)

V(i,0) = 0

V(i,M) = S\_max - Ke^(-r\*(i∆t))

**Solution**

To solve, you have to start solving for unknowns backward in time (starting from maturity time T), since the boundary conditions are known for V(n,j) = max(j∆S – K ,0). In that case you have three unknowns and one knows variable give the equation:

So we need to use Matrix Algebra to solve for the unknown values.

We can rewrite in matrix form:

We can further write as:

Then we can solve:

**Forward Difference (Backward Euler)**

I prefer a different approach however for implementing. Here it is.

We start with the PDE

**We substitute:**

**Therefore:**

So we need to use Matrix Algebra to solve for the unknown values.

We can rewrite in matrix form:

We can further write as:

Then we can solve:

**Backward Difference**

* For first spot derivative use central difference
* For second spot derivative use

Moreover

* dS = ∆S
* dt = ∆t
* M∆S = S\_max
* N∆t = T
* V(i,j) denotes V(i∆t, j∆S)

BS PDE

Note, that this becomes the Forward Euler using the Backward Difference which can be confusing, to say the least! This is also called Explicit Euler, since this is an explicit formula for . The Explicit method can be very unstable! Be Careful! If you make dS step small, you have to make dt step very small! But you have to make dS step relatively small, otherwise you will have poor accuracy, but this makes the computation very expensive.

**Boundary Conditions**

V(n,j) = max(j∆S – K ,0)

V(i,0) = 0

V(i,M) = S\_max - Ke^(-r\*(i∆t))

**Solution**

Here we can solve with simple loop as the unknown variable at time i-1 is a result of known variables at time I, and since we start solving backwards in time (since the boundary condition is knows as T goes to maturity), we can start solving with a simple loop, no need for matrix notation.

**IMPORTANT!!!**

We have stability problem with Backward Euler in this case if becomes less than 0.5. This is similar to the properties of Forward Euler in Heat Equation PDE.

**Crank Nicolson**

Here we will take weighted average of forward and backward difference.

Forward difference:

Lets deduct td :

Backward Difference:

So now we take weighted average of both:

T from one side, t-dt from the other side:

So…

So…

So…

We solve via matrix

Then we can solve:

Or to be more easier for programming:

Change of Variables

We have Black Scholes PDE:

We want to run it in log terms:

Then we have, by the chain rule:

Therefore, PDE becomes:

**Generic SABR PDE**

**Generic SABR SDE**

**Generic SABR PDE**

Therefore, we have

Hedged portfolio (hedged on F and sigma – two sources of uncertainties)

Replacing dV:

**Generic SABR PDE:**

Removing dt:

**Generic PDE in log:**

Define:

Therefore

Then we have, by the chain rule:

For Forward F:

For sigma :

For the intersection of F and sigma:

Now we substitute:

**Generic SABR PDE in log:**

**Hagan SABR PDE**

**Hagan SABR SDE**

**Hagan SABR PDE**

Substituting in the below:

More clearly:

**Hagan SABR PDE in log**

Define:

In the specific case of B=1, F disappears.

We get W. McGhee PDE from paper:

<https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3288882>

**ZABR PDE**

**ZABR SDE**

With , we get Hagan SABR

**ZABR PDE**

Substituting in the below:

More clearly:

This is the same PDE equation we get in Implementing ZABR paper by Peter Caspers:

<https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2692048>

**ZABR PDE in log**

Define:

**ZABR PDE (F in spot, vol in log)**

**Finite Difference Scheme SABR**

**Forward Difference**

**Implicit Euluer**

* For time derivative
* For first spot derivative use central difference
* For second spot derivative use
* For first vol derivative use central difference
* For second vol derivative use
* For the mixed spot/vol derivative:

Or if we use the other scheme

Moreover

* dF = ∆F
* dt = ∆t
* M∆F = F\_max
* N∆t = T
* K∆ =

Therefore,

We have from earlier:

So it becomes:

**OK, so it appears we have to use the other scheme for the mixed derivative…**

Therefore,

We have from earlier:

So it becomes:

**Boundary Conditions**

We consider Forward from [0, infinity] , Volatility from [0, infinity], Time from [0,T]

At expiry t=T

At Forward=0 (or minus 2?)

At Forward = Infinity (F\_max)

0)

At Sigma = Infinity (Sigma\_max)

**Solution**

To solve, you have to start solving for unknowns backward in time (starting from maturity time T), since the boundary conditions are known for V(n,j) = max(j∆S – K ,0). In that case you have three unknowns and one known variable give the equation:

So we need to use Matrix Algebra to solve for the unknown values.

We can rewrite in matrix form:

We can further write as:

Then we can solve:

**Appendix**

Finite difference for mixed spot vol derivative:

Taylor series

Add first two:

Add last four:

Deduct last two equations:

Therefore

Another way…

Finite difference for mixed spot vol derivative:

Taylor series

Add first two:

Add last two:

Deduct last two equations:

Therefore

**Fokker Plank PDE**

**Kolmogorov Equation**

The Theory of Kolmogorov: the PDE of the Probability Density Function

Usually of the form:

Once you solve the PDE, therefore you know:

you can then solve for option prices such as:

Which in practice is done is going through each element in the PDE grid, where each element gives you

So the theory tells you that, if you have the SDE:

And you have some expectations of y(t):

Then Kolmogorov PDE is :

is the PDF

**Black Scholes Fokker Plank PDE**

We have the SDE:

In log:

Then the Fokker Plank PDE is :

PDE in log:

**SABR Fokker Plank PDE**

We have the SDE:

Then the Fokker Plank PDE:

So there must be a clarification here. Important one!!!!

The time in the above equation is actually . Otherwise the equation will be a change of sign.

This coincides with Hagan PDE equation from:

<https://www.researchgate.net/publication/264718376_Arbitrage-free_SABR>

Define:

Therefore

Fokker Plank PDE in log:

Fokker Plank PDE mixed (normal Spot, log in vol):

**Dupire Local Volatility PDE**

Dupire SDE

Dupire PDE

Dupire PDE in Log

**Generic PDE 1D Solver**

1. Implicit Euler
2. Crank Nicolson

In Matrix Form, we have the following:

* General case:

Lower Matrix is :

Upper Matrix is :

Diagonal Matrix is :

Rhs:

* For the Boundaries:

Rhs:

At x=1:

In this case T+1, x-1 coefficients and Grid values are the left boundary values, so they are always available.

At x=n-1:

In this case T+1, x+1 coefficients and Grid values are the right boundary values, so they are always available.